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STABILITY AND CONTROL ANALYSIS OF AN UNEMPLOYMENT MODEL INCORPORATING INTEGRATION PROGRAMS EFFECT

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Abstract. A new mathematical model is proposed to investigate the mechanisms of unemployment with government interventions such as internships programs. The proposed model consists of a compartmental system of four components, which are the unemployment U, the employment E, the unemployed beneficiaries of government programs U_p and the cumulative density of programs P. The qualitative analysis of differential equations is used to analyze the model. First the equilibrium's states are calculated, then the analysis of their stability is discussed with to help of Routh-Hurwitz criterion and Lyapunov's method. Furthermore,in order to achieve the effectiveness and performance of government programs an optimal control strategy is introduced, in which two control variables are introduced. Finally, to validate the main analytical results, some numerical investigations are performed. Our findings suggest that, a government's efforts to increase employment opportunities and assist unemployed people in starting their own businesses are more effective at lowering the unemployment rate. Further, companies should be encouraged to take on the unemployed who have benefited from internship programs.

Keywords: Mathematical modelling, unemployment, integration programs, stability, optimal control.
AMS Subject Classification: 97M10.
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1 Introduction

Unemployment is a problem for both developing and developed countries, and it may negatively affect people's lives.Unemployment is a major socio-economic issue in many countries, and understanding the factors behind it is crucial to developing effective policies to reduce unemployment and promote employment. Unemployment is when people of all levels of education are looking for work, but can't find anything they're interested (Okojie, 2003). The inadequacy of training for the labour market and the current technological revolution are the main factors increasing unemployment. In addition to these fundamental factors which increase the number of unemployed, job losses can be caused by economic crises, natural disaster,and health issues which further increase the number of unemployed.

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Unemployment is a problem that can affect a nation's economy, society, and politics. People without jobs are more prone to mental and psychological problems like depression, substance abuse, suicide, and heart disease (Eisenberg & Lazarsfeld, 1938; Murphy & Athanasou, 1999). Therefore the governments are focusing their economic and social efforts on the sustainable development by implementing programs and strategies to combat unemployment Lately, scientific researchs based on mathematical models to analyse the problem of unemployment has been the subject of many studies by several authors Misra & Singh (2011, 2013); Munoli et al. (2017); Al-Maalwi et al. (2018); El Yahyaoui & Amine (2022). These authors have addressed the problem of unemployment in a scientific manner in order to point the way to the development of effective measures to combat it. Based of these scientific approaches, the interest of research in this domain has increased. Some authors have studied the dynamics of unemployment from a several angles, identifying the impact of public policies and actions on controlling and reducing this social phenomenon (Misra et al., 2017; El Hammioui, 2020; Pletscher, 2015; Sanghi & Srija, 2015; Singh, 2020; Singh et al., 2023; Njike-Tchaptchet & Tadmon, 2023; Daud & Ghozali, 2015; Chinnadurai & Athithan, 2022).

It should be noted that Misra and Singh provided the first mathematical models defining the unemployment problem in 2011. This work being motivated by the work Nikolopoulos & Tzanetis (2003). In Pathan (2015) the authors studied the way in which self-employment influenced the unemployment rate. They determined that reducing unemployment could result by both creating new job opportunities and individuals transitioning into self-employment. In Daud & Ghozali (2015) a simplified mathematical model is developed based only of the amount of people in employment and the number of people without employment as two distinct variables.

In Pathan & Bhathawala (2017) the authors approached the study of unemployment differently by introducing a new variable related to the influx of immigrants. Therefore, they suggested that an increase in the number of migrant workers allowed into the regions would require a corresponding increase in job opportunities relative to the local unemployment rate. Moreover, in Misra et al. (2017) studied the importance of training offered by specific institutions to the unemployed as a crucial element in solving the problem of unemployment. The objective of this program is to prepare participants for temporary and independent work. The authors showed that as the training programs were successful, there were fewer unemployed and more self-employed and temporary workers.

In Singh (2020) the authors point the importance of highly skilled workers is highlighted by the authors as a potential force for decreasing unemployment by attracting other job seekers. In addition, they evaluated the cost of acquiring highly qualified employees and the number of jobs available for them. In Ashi et al. (2022) the authors evaluated how government aid can reduce unemployment in developed nations.

In El Yahyaoui & Amine (2022) for the first time a new modelling methodology is proposed based on the consideration of the components of unemployment. This approach has the potential to assist government authorities in simulating the consequences of targeted economic measures designed to revive those temporarily unemployed due to economic cycles and prevent them from transitioning into long-term structural unemployment. In Njike-Tchaptchet & Tadmon (2023) the authors studied the problem of unemployment in the presence of a financial crisis and show that it is the most effective way for a governments to help the unemployed to create their own activity, which will enable them to generate new jobs, rather than to help the self-employed to create new job opportunities.

Following an investigation of the literature presented above, we see the need for a contribution focusing on collaboration between government and business to tackle unemployment. the purpose of this study is to show that internships is a central element of the overall strategy to reduce unemployment. It's commonly known that Internships offers to individuals the opportunity to acquire specific practical and professional skills which can lead to unemployment reduction.

In this context, the government intervention aim speeds up the process of matching the unemployed population with the labour market through work placement programs and by supporting firms in reintegrating the unemployed who have completed work placements. This orientation, should involve the introduction of incentive policies based on fiscal instruments to facilitate young people's access to the labour market.

In the current study, we created a mathematical model to examine the contribution of integration programs to lowering unemployment. Our work differs from previous studies of the links between skills development and unemployment as we take into consideration the distinctive abilities of without unemployment , and highlights a new governmental approach that supports companies taking part in internship programs. To this end, our contribution to unemployment research is as following:

(i) Formulation and mathematical analysis of a new model describing the dynamics of unemployment by introducing into the active population an intermediate class between the class of job seekers and the class of employed people. This class is made up of people who have benefited from government work placement programs.

(ii) Design and resolution of an optimal control strategy that public authorities may find useful in defining and implementing policies aimed for lowering the rate of unemployment.

(iii) Numerical illustration that validate theoretical findings.

The paper consists of the following sections: The description and formulation of the study are presented and explained in section 2. The model equilibrium is examined and established in section 3. The model stability study is presented and discussed in section 4. A numerical simulations are shown and discussed in section 5 to support the theoretical approach. In order to guarantee the effectiveness of the proposed programs, in section 6 we formulate and solve an optimal control of the unemployment problem.

2 Model formulation

In the modelling process at any time t, the active population is N(t) We assume that in the region under study there are three classes of the total active population N(t). The number of without employment persons denoted $U(t)$, the number of employed individuals $E(t)$ and the class of the unemployed beneficiaries of government programs $U_p(t)$. Let $P(t)$ represent the total density of internship programs implemented by the government at time t. It is assumed that the increase in the amount of program density is considered to be positively correlated to the number of workers people. It is considered that, due to government programs to combat unemployment, the unemployed form a different and distinct class from the employed class.

Thus, the following nonlinear differential system governs the model's dynamics:

$$\begin{aligned}
\frac{dU(t)}{dt} &= A - \beta U(t)E(t) - \lambda U(t)P(t) + \gamma E(t) + \lambda_0 U_p(t) - dU(t),\\
\frac{dE(t)}{dt} &= \beta U(t)E(t) - \gamma E(t) + aU_p(t) - \alpha E(t) - dE(t),\\
\frac{dU_p(t)}{dt} &= \lambda U(t)P(t) - \lambda_0 U_p(t) - aU_p(t) - dU_p(t),\\
\frac{dP(t)}{dt} &= \mu E(t) - \mu_0 P(t),
\end{aligned} \tag{1}$$

where $U(0) \geq 0, E(0) \geq 0, U_p(0) \geq 0, P(0) \geq 0$.

The model's parameters are positive constants with the following definitions:

A is the recruitment unemployed rate;

β is the transfer rate at which people move from the without work to the employed class;

λ represents the distribution rate of internship programs among the unemployed people, which classifies them into a different class;
γ is the rate of workers people who lose their employment and become jobless;
λ_0 designates the transfer rate of people who have taken part in work placement programmes but are having difficulty re-entering the labour market and are returning to the ranks of the unemployed;
a is the transfer rate from the class of people benefiting from internship programs to the class employed people;
d refers to the mortality or migration rate;
μ indicates the rate at which beneficial programs are put into implementation;
μ_0 is the rate of exhaustion of integration programs due to their ineffectiveness, caused by social and economic problems.

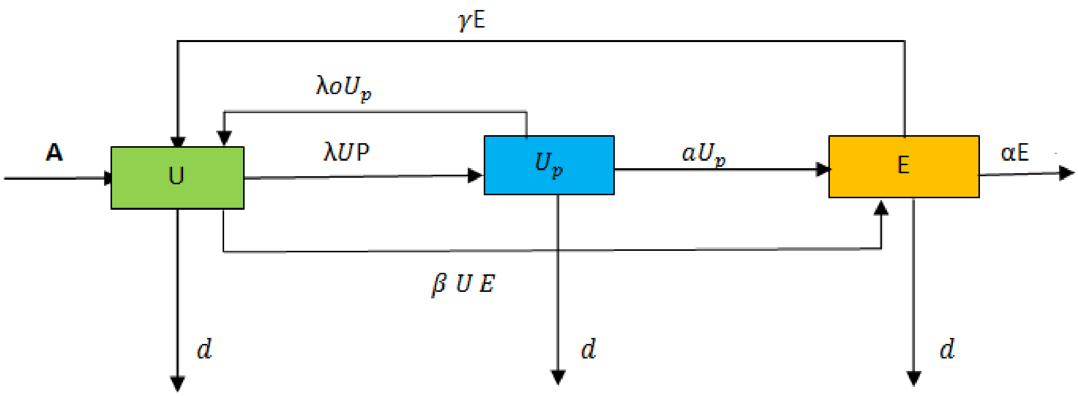

Figure 1: The flowchart of unemployment model (1)

Since $U + E + U_p = N$, the model system 1 reduces to the system below:

$$\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}t} &= \beta\left(N - E - U_p\right)E - \gamma E - \alpha E - dE + aU_p, \\
\frac{\mathrm{d}U_p}{\mathrm{d}t} &= \lambda\left(N - E - U_p\right)P - dU_p - \lambda_0 U_p - aU_p, \\
\frac{\mathrm{d}N}{\mathrm{d}t} &= A - dN - \alpha E, \\
\frac{\mathrm{d}P}{\mathrm{d}t} &= \mu E - \mu_0 P.
\end{aligned} \tag{2}$$

Now we just need to study model system 2 in detail,instead of model system 1. For the study of model 2, We determined a region of attraction provided by the set:
If $(E, U_p, N, P) \in \mathbb{R}^4_+$, then the set defined by

$$\Omega = \left\{(E, U_p, N, P) : U_p \leq N \leq \frac{A}{d}, 0 \leq P \leq \frac{\mu A}{\mu_0 d}, 0 \leq E\right\}$$

is bounded and positively invariant.

3 Equilibrium studies

There are two non-negative equilibria for the model system 2 given below:

(i) $Q_1^\star\left(0,0,\frac{A}{d},0\right)$;

(ii) $Q_2^\star\left(E^*,U_p^*,N^*,P^*\right)$.

The equilibrium $Q_1^\star$ is easily obtained when $E=0$. The components of the equilibrium point $Q_2^\star$ are solutions of the following algebraic equations (for $E\neq 0$)

$$\begin{aligned}\beta\left(N-E-U_p\right)E-\gamma E-\alpha E-dE+aU_p &= 0, &(3)\\ \lambda\left(N-E-U_p\right)P-dU_p-\lambda_0 U_p-aU_p &= 0, &(4)\\ A-dN-\alpha E &= 0, &(5)\\ \mu E-\mu_0 P &= 0. &(6)\end{aligned}$$

From (3) and(6), we get

$$U_p^*=\frac{\lambda\mu\left(d+\gamma+\alpha\right)}{\lambda a\mu+\beta\mu_0\left(d+\lambda_0+a\right)}E^* \tag{7}$$

and

$$P^*=\frac{\mu E^*}{\mu_0}. \tag{8}$$

From equation (5), we have

$$N^*=\frac{\left[A-\alpha E^*\right]}{d}, \qquad \textit{where} \quad A>\alpha E^*. \tag{9}$$

Substituting (7) and (9) in (3), we obtain

$$\beta\left(\frac{A-\alpha E^*}{d}-E^*-\frac{\lambda\mu\left(\gamma+d+\alpha\right)}{\lambda a\mu+\beta\mu_0\left(d+\lambda_0+a\right)}E^*\right)=\gamma+d+\alpha-a\frac{\lambda\mu\left(\gamma+d+\alpha\right)}{\lambda a\mu+\beta\mu_0\left(d+\lambda_0+d+a\right)}. \tag{10}$$

This yields the value of E^* as

$$E^*=\frac{\left(d+\lambda_0+a\right)\mu_0[\beta A-d\left(d+\gamma+\alpha\right)]+aA\lambda\mu}{\left(\alpha+d\right)\left[\lambda a\mu+\beta\mu_0\left(d+\lambda_0+a\right)\right]+d\lambda\mu\left(d+\gamma+\alpha\right)}. \tag{11}$$

It is clear that $Q_2^\star$ will only exist when E^* is positive, i.e

$$\frac{\beta A}{d}-\left(\gamma+d+\alpha\right)>0,$$

$$R_0=\frac{\beta A}{d\left(d+\gamma+\alpha\right)}>1.$$

R_0 can be defined as the system (1)'s imput that can ensure the existence of an equilibrium $Q_2^\star$, when $R_0>1$.

4 Stability analysis

4.1 Local stability analysis

In this section we investigate the local stability of system (2) around each of the equilibrium points by calculating the matrix of variations $V(Q^\star)$ where $Q^\star$ is an equilibrium point. The stability conditions required for equilibrium points $Q_1^\star\left(0,0,\frac{A}{d},0\right)$ and $Q_2^\star\left(E^*,U_p^*,N^*,P^*\right)$ are presented below.

The Jacobian matrix V($Q_1^\star$) of system (2) at the positive equilibrium $Q_1^\star$ is given by

$$V(Q_1^\star)=\begin{bmatrix}\frac{\beta A}{d}\left(d+\gamma+\alpha\right) & a & 0 & 0\\ 0 & -\left(d+\gamma+a\right) & 0 & \frac{\lambda A}{d}\\ -\alpha & 0 & -d & 0\\ \mu & 0 & 0 & -\mu_0\end{bmatrix}.$$

Theorem 1. *The equilibrium $Q_1^\star\left(0,0,\frac{A}{d},0\right)$ is locally asymptotically stable if $\frac{\beta A}{d}(d+\gamma+\alpha)<0$, that is, if $R_0<1$ and gets unstable if $R_0>1$*

The Jacobian matrix V($Q_2^\star$) of system (2) at the positive equilibrium $Q_2^\star$ is given by

$$V(Q_2^\star)=\begin{bmatrix} -p_1 & -p_2 & p_3 & 0\\ p_4 & -p_5 & p_6 & -p_7\\ -p_8 & 0 & -p_9 & 0\\ p_{10} & 0 & 0 & -p_{11}\end{bmatrix},$$

where $p_1=\frac{aU_p^\star}{E^\star}+2\beta E^\star$, $p_2=\beta E^\star-a$, $p_3=\beta E^\star$, $p_4=-\lambda P^\star$, $p_5=\lambda P^\star+(d+\lambda_0 a_3+a)$, $p_6=\lambda P^\star$, $p_7=\lambda\left(N^\star-E^\star-U_p^\star\right)$, $p_8=\alpha$, $p_9=d$, $p_{10}=\mu_0$ and $p_{11}=\mu$.

The matrix above has the following characteristic equation:

$$\lambda^4+K_1\lambda^3+K_2\lambda^2+K_3\lambda+K_4=0, \tag{12}$$

$$\begin{aligned}K_1=&p_1+p_5+p_9+p_{11},\\ K_2=&p_2p_4+p_1p_5+p_3p_8+p_1p_9+p_5p_9+p_1p_{11}+p_5p_{11}+p_9p_{11},\\ K_3=&p_3p_5p_8-p_2p_6p_8+p_2p_4p_9+p_1p_5p_9+p_2p_7p_{10}+p_2p_4p_{11}\\ &+p_1p_5p_{11}+p_3p_8p_{11}+p_1p_9p_{11}+p_5p_9p_{11},\\ K_4=&p_2p_7p_9p_{10}-p_3p_5p_8p_{11}+p_2p_6p_8p_{11}+p_2p_4p_9p_{11}+p_1p_5p_9p_{11}.\end{aligned}$$

Since K_1 and K_4 are positive and some simple algebraic operations yield the following: $K_1K_2-K_3>0$ and $K_3\left(K_1K_2-K_3\right)-K_1^2K_4>0$. Applying the Routh-Hurwitz criteria, the roots of the above matrix's characteristic equation are either negative or have a negative real part. As a result, local stability is established.

4.2 Global stability analysis

Theorem 2. *The equilibrium $Q_2^\star\left(E^\star,U_p^*,N^*,P^*\right)$ is globally asymptotically stable if the following conditions are satisfied:*

(i) $(\beta E^\star)^2<\frac{4}{9}\left(-\beta N^*+2\beta E^\star+\beta U_p^*+d+\gamma+\alpha\right)m_1\left(\mu P^*+d+\lambda_0+a\right)$,

(ii) $\frac{K\mu^2\left(N^*-E^\star-U_p^*\right)^2 a}{(\lambda P^*+d+\lambda_0+a)^2\frac{2\beta E^\star}{3\alpha a}}<\frac{1}{3}\left(-\beta N^*+2\beta E^\star+\beta U_p^*+d+\alpha+\gamma\right)\mu_0$.

Proof. To check global stability of equilibrium point $Q_2^\star$, we use Lyapunov's method. Then we consider a positive definite function U such that

$$U=\frac{1}{2}\left(E-E^\star\right)^2+\frac{1}{2}m_1\left(U_p-U_p^\star\right)^2+\frac{1}{2}m_2\left(N-N\star\right)^2+\frac{1}{2}m_3\left(P-P^\star\right)^2, \tag{13}$$

where the coefficients m_1, m_2 and m_3 are some positive constants to be determined bellow. On differentiating U with respect to t, we get

$$\frac{dU}{dt}=\left(E-E^\star\right)\frac{dE}{dt}+m_1\left(U_p-U_p^\star\right)\frac{dU_p}{dt}+m_2\left(N-N^\star\right)\frac{dN}{dt}+m_3\left(P-P^\star\right)\frac{dP}{dt}$$

which due to (1), (3), (4), (5) and (6) gives

$$\begin{aligned}\frac{dU}{dt}=&\left(E-E^\star\right)\left[\beta\left(N-E-U_p\right)E-\gamma E-\alpha E-dE+aU_p\right]+m_1\left(U_p-U_p^\star\right)\left[\lambda\left(N-E-U_p\right)P\right.\\ &\left.-\left(d+\lambda_0+a\right)U_p\right]+m_2\left(N-N^\star\right)\left[A-dN-\alpha E\right]+m_3\left(M-M^\star\right)\left[\mu_0E-\gamma P\right]\\ =&\left(E-E^\star\right)\left[\beta\left(N^*-E^\star-U_p^*\right)E^\star-(d+\lambda_0+a)E^\star+aU_p^*+\beta\left(N^*-E^\star-U_p^*\right)\left(E-E^\star\right)\right.\\ &-(d+\lambda_0+a)\left(E-E^\star\right)+aU_p+\beta\left(\left(N-N^\star\right)-\left(E-E^\star\right)-U_p\right)\left(E-E^\star\right)+\beta\left(\left(N-N^\star\right)-\left(E-E^\star\right)-U_p\right)E^\star\left.\right]\\ &+m_1\left(U_p-U_p^\star\right)\left[\lambda\left(N^*-E^\star-U_p^*\right)P^*-\left(d+\lambda_0+a\right)U_p^*+\lambda\left(n-i-U_p\right)P\right.\\ &\left.-\left(d+\lambda_0+a\right)U_p+\lambda\left(N^*-E^\star-U_p^*\right)\left(P-P^*\right)+\lambda\left(\left(N-N^\star\right)-\left(N-N^\star\right)-\left(U_p-U_p^\star\right)\right)P^*\right]\\ &+m_2\left(N-N^\star\right)\left[A-dN^*-\alpha E^\star-d\left(N-N^\star\right)-\alpha\left(E-E^\star\right)\right]\\ &+m_3\left(P-P^\star\right)\left[\mu E^\star-\mu_0P^*+\mu\left(E-E^\star\right)-\mu_0\left(P-P^\star\right)\right].\end{aligned}$$

On linearizing the above equation, we get

$$\begin{aligned}
\frac{dU}{dt} =& (E-E^\star)\left[\beta\left(N^*-E^\star-U_p^*\right)(E-E^\star)-(d+\alpha+\gamma)(E-E^\star)+a\left(U_p-U_p^\star\right)\right.\\
&+\beta\left((N-N^\star)-(E-E^\star)-\left(U_p-U_p^\star\right)\right)(E-E^\star)\\
&\left.+\beta\left((N-N^\star)-(E-E^\star)-\left(U_p-U_p^\star\right)\right)E^\star\right]\\
&+m_1\left(U_p-U_p^\star\right)\left[\lambda\left((N-N^\star)-(E-E^\star)-\left(U_p-U_p^\star\right)\right)P^*\right]-(d+\lambda_0+a)\left(U_p-U_p^\star\right)\\
&+\lambda\left(N^*-E^\star-U_p^*\right)(M-M^\star)+\lambda\left((N-N^\star)-(E-E^\star)-\left(U_p-U_p^\star\right)\right)P^*\Big]\\
&+m_2(N-N^\star)\left[d(N-N^\star)-\alpha(E-E^\star)\right]+m_3(M-M^\star)\left[\Lambda(E-E^\star)-\mu_0(M-M^\star)\right]\\
=&-(E-E^\star)^2\left[-\beta\left(N^*-E^\star-U_p^*\right)+\beta E^\star+(d+\alpha+\gamma)\right]-\left(U_p-U_p^\star\right)^2 m_1\left[(d+\lambda_0+a)+\lambda P^*\right]\\
&-(N-N^\star)^2\left[m_2 d\right]-(P-P^\star)^2\left[m_3\mu_0\right]+(E-E^\star)\left(U_p-U_p^\star\right)\left[a-\beta E^\star-m_1\lambda P^*\right]\\
&+(E-E^\star)(N-N^\star)\left[\beta E^\star-m_2\alpha\right]+(E-E^\star)(P-P^\star)\left[-m_3\mu\right]\\
&+\left(U_p-U_p^\star\right)(P-P^\star)\left[m_1\lambda\left(N^*-E^\star-U_p^*\right)\right]+\left(U_p-U_p^\star\right)(N-N^\star)\left[m_1\lambda P^*\right].
\end{aligned}$$

$\frac{dU}{dt}$ will be negative definite provided, for $k>\frac{3}{2}$,

$$\begin{gathered}
\left\{a-\beta E^\star-m_1\lambda\left(P^*\right)\right\}^2<\frac{4}{9}\left(-\beta N^*+2\beta E^\star+\beta U_p^*+d+\gamma+\alpha\right)m_1\left(\mu P^*+d+\lambda_0+a\right),\\
\left(\beta E^\star-m_2\alpha\right)^2<\frac{2}{3}\left(-\beta N^*+2\beta E^\star+\beta U_p^*+d+\alpha+\gamma\right)m_2 d,\\
m_3\mu^2<\frac{1}{3}\left(-\beta N^*+2\beta E^\star+\beta U_p^*+d+\alpha+\gamma\right)\mu_0,\\
m_1\mu_0^2\left(N^*-E^\star-U_p^*\right)^2<\frac{2}{3}m_3\left(\lambda P^*+d+\lambda_0+a\right)\mu_0,\\
m_1\lambda^2\left(P^*\right)^2<\frac{2}{3}m_2\left(\lambda P^*+d+\lambda_0+a\right)d
\end{gathered}$$

On choosing

$$m_2=\frac{\beta E^\star}{\alpha}$$

$$m_1=\frac{a}{\lambda M*}$$

and

$$m_3=\frac{K\lambda^2\left(N^*-E^\star-U_p^*\right)^2 a}{\left(\lambda P^*+d+\lambda_0+a\right)\mu_0\lambda P^*},$$

for $k>\frac{3}{2}$. □

5 Numerical simulations

In this section, we validate the theoretical results with numerical illustrations. which are based on the simulation of the solutions of the mathematical model 2. Our tests aim to verify the presence of the equilibrium point $Q_2^\star\left(E^\star,U_p^\star,N^\star,P^\star\right)$ and its stability conditions as well as to show the impact of significant factors on unemployment and employment in accordance with their rate of change. We carried out several numerical calculations using MATHLAB, choosing the following parameter values: $A=1200$, $\beta=0.446$, $d=0.35$, $\gamma=0.0035$, $\alpha=0.40$, $a=0.00005$, $\lambda=0.65$, $\lambda_0=0.3020$, $mu=0.490$, $\mu_0=0.016$.

For the above set of parameter, the condition of existence of equilibrium $E_2\,(i.e.R_0>1)$ and the stability condition in theorem 2 are satisfied. The equilibrium values for the model is $E^\star=63.52$, $U_p^\star=3292.97$ $N^\star=3358.2$ and $P^\star=2676.96$. Then we find that all eigenvalues of the variational matrix corresponding to the equilibrium $Q_1^\star\left(E^\star,U_p^\star,N^\star,P^\star\right)$ for the model system 2 are negative,so the equilibrium $Q_1^\star\left(E^\star,U_p^\star,N^\star,P^\star\right)$is locally asymptotically stable Similarly we can note that for all of the above parameter values, all eigenvalues of the variational matrix corresponding to the equilibrium $Q_2^\star\left(E^\star, U_p^\star, N^\star, P^\star\right)$ for the model system 2 are negative so the equilibrium $Q_2^\star\left(E^\star, U_p^\star, N^\star, P^\star\right)$is locally asymptotically stable.

Different simulations are also performed for various initial values and the results are illustrated in figures 2-5 . These figures show respectively that the numerical solution of unemployment people $E(t)$, the unemployment people beneficiaries of government programs $U_p(t)$, the unemployment people $U(t)$ and cumulative density of programs driven by the government $P(t)$ for different initial values. We observe that all numerical solutions approach their equilibrium points for different choices of initial conditions. Hence we infer that the system is globally asymptotically stable in the region of attraction around equilibrium point $Q_2^\star\left(E^\star, U_p^\star, N^\star, P^\star\right)$ for the above set of parameters.

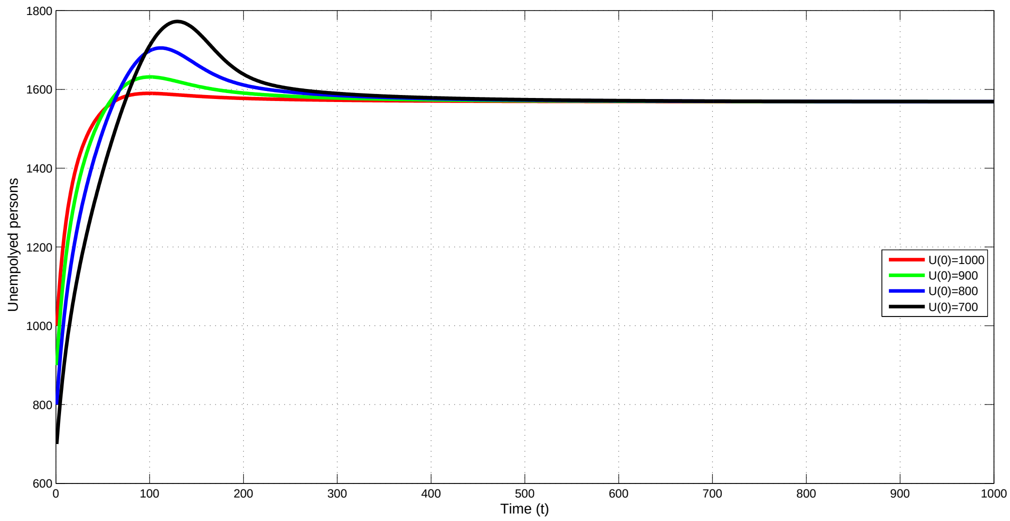

Figure 2: Numerical simulation of unemployment for different starting conditions $U(0)$.

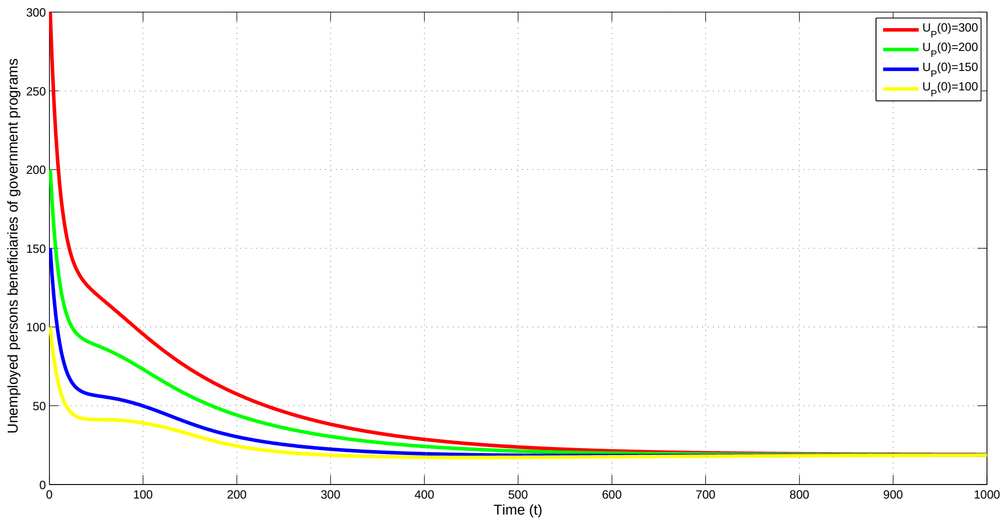

Figure 3: Numerical simulation of unemployment beneficiaries of government programs for different starting conditions $U_p(0)$.

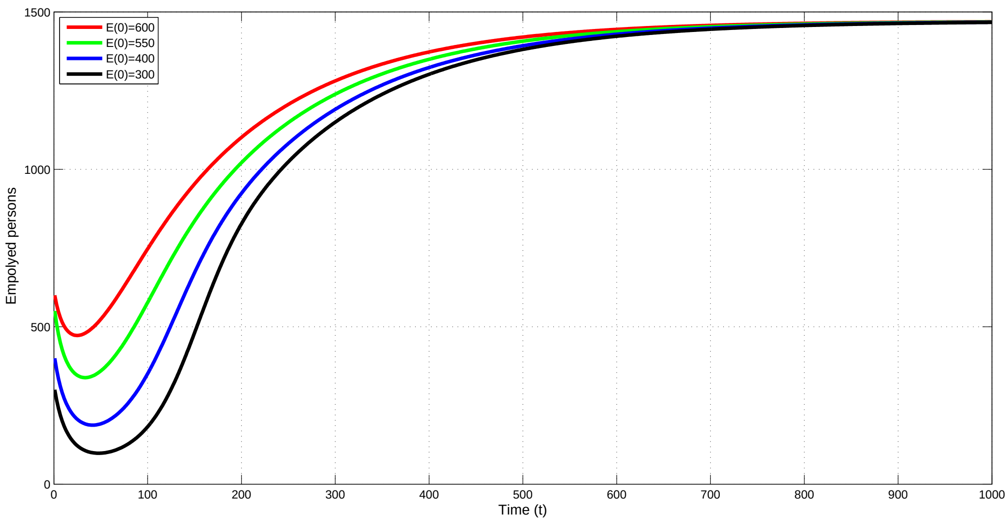

Figure 4: Numerical simulation of employment for different starting conditions $E(0)$.

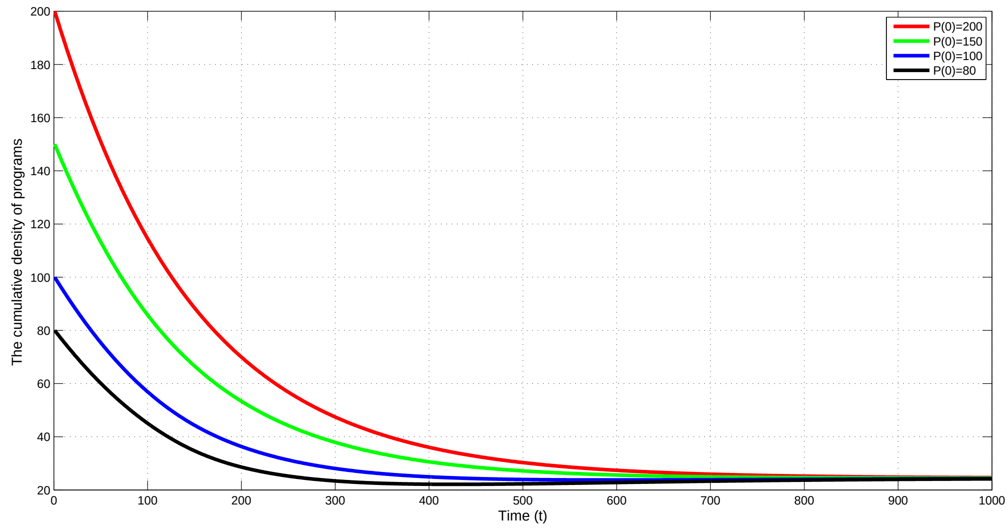

Figure 5: Numerical simulation of cumulative density of programs driven by the government for different starting conditions $P(0)$.

Furthermore, we observe the impact of the different rate a on the unemployment, on the employment and on proportion of unemployment people beneficiaries of internships programs, we remark that when a increases, the number unemployment decreases Fig 6. Unlike in Fig 7, we can see that when a increases, the number of employed increases. While the Fig 8 shows that the increase in the rate of transition of qualified people to the employed category explains the importance of internship programs in the creation of more jobs.

Figures 2, 3, 4 and 5 show respectively that the numerical simulation of unemployment people $U(t)$, employment people $E(t)$, unemployment people beneficiaries of government programs $U_p(t)$, and cumulative density of programs driven by the government $P(t)$ for the different initial values. Thus, we observe that the curves of solution converge to the unique equilibrium point. 4.2.

Figures 6 and 7 show respectively that when the transfer rate from the class of people benefit-

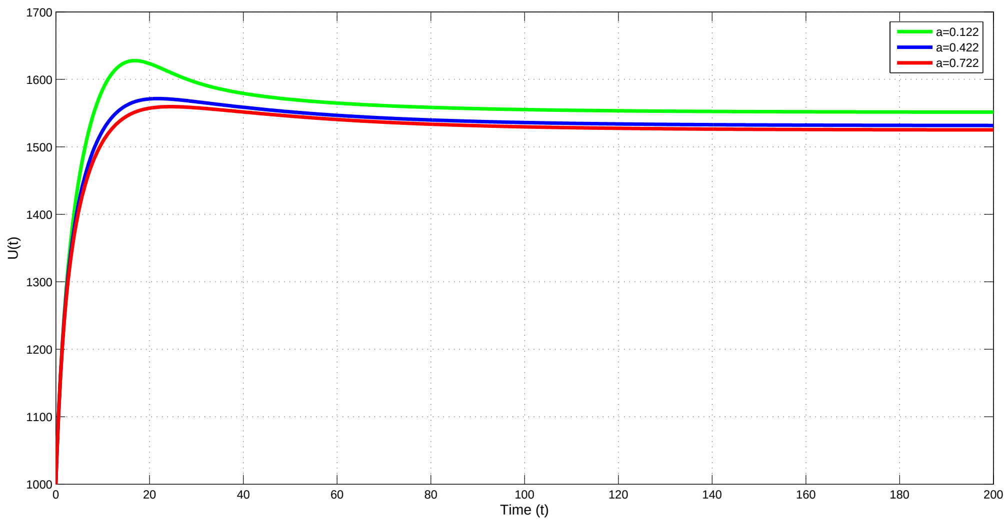

Figure 6: Variation unemployment numbers through time for various values of a.

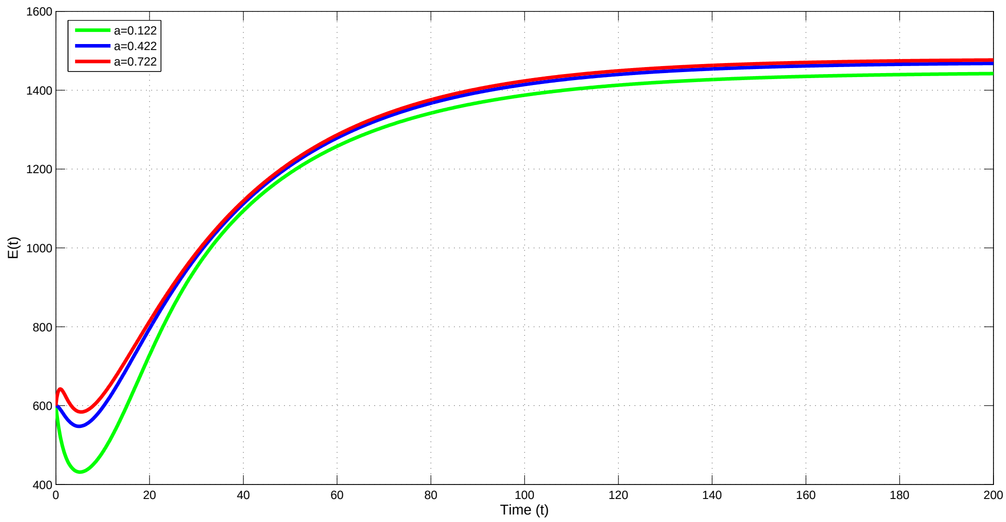

Figure 7: Variation employment numbers through time for various values of a.

ing from internship programs to the class employed people increases,the number unemployment decreases,the number of employed increases,while 8 shows that the increase in this rate indicates that more trainees are entering employment, which reflects the effectiveness of the integration programs.

6 The optimal control of the unemployment problem

6.1 Formulation of the optimal control problem

The objective of the proposed control strategy is to minimize the number of unemployed people $U(t)$ and the number of unemployed beneficiaries of government programs $U_P(t)$ by increasing the rate of reintegration of trainees into the labour market and to maximize the number of the employed people $E(t)$ during the time step k $=0$ to k $=t_f$. The social and economic problems caused by this phenomenon will also be reduced. In order to achieve these objectives,

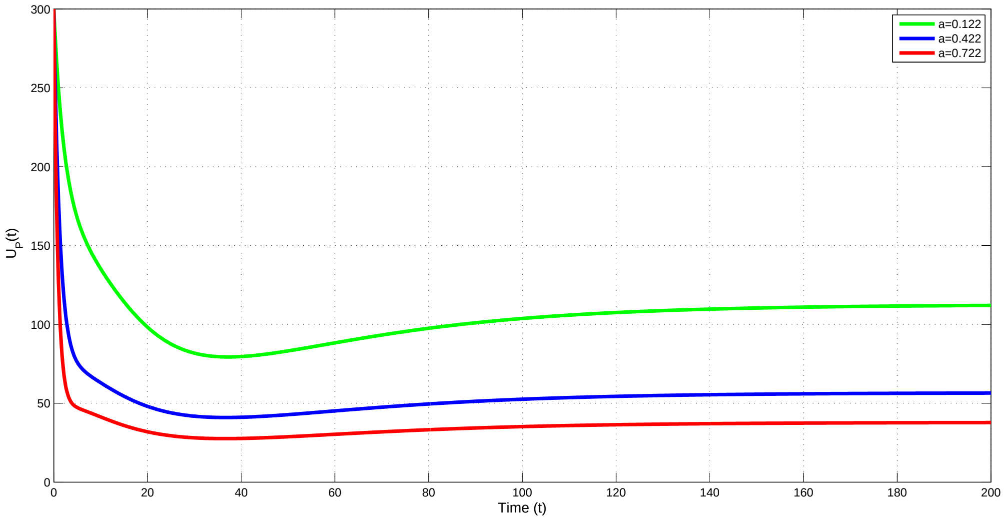

Figure 8: Variation unemployed beneficiaries of government program through time for various values of a.

we introduce two control variables. The first control u represents the efforts of government programs to create more employment opportunities and to encourage people to develop personal activities. The second control v represent the measures for supporting and integrating trainees, by encouraging companies to employ people benefiting from trainee-ships.
The controlled mathematical system is given by the following system of difference equation:

$$\begin{cases} \frac{dU(t)}{dt} &= A - \beta \mathbf{u(t)})U(t)E(t) - \lambda U(t)P(t) + \gamma E(t) + \lambda_0 U_P(t) - dU(t). \\ \frac{dE(t)}{dt} &= \beta(1 + \mathbf{u(t)}U(t)E(t) - \gamma E(t) + aU_P(t) - \alpha E(t) - dE(t) + \mathbf{v(t)}U_P(t). \\ \frac{dU_P(t)}{dt} &= \lambda U(t)P(t) - \lambda_0 U_P(t) - aU_P(t) - dU_P(t) - \mathbf{v(t)}U_P(t). \\ \frac{dP(t)}{dt} &= \mu E(t) - \mu_0 P(t). \end{cases} \tag{14}$$

6.2 The optimal control: existence and characterization

Optimal control theory, with a history of successful applications in biological and medical problems, is recently used to modelling social problems (Arora & Yadav, 2023; Munoli et al., 2017; Mallick & Biswas, 2017, 2020; Kamien & Schwartz, 2012; Bentaleb et al., 2023; Bouajaji et al., 2021). Using this approach, the aim is to identify the optimal strategy needed to reduce unemployment and achieve the efficiency and effectiveness of government programs to create more jobs and help the unemployed integrate into the labour market. Then for this objective we consider the control variable $(u(t), v(t)) \in \Omega_{ad}$

$$\begin{aligned} &\Omega_{ad} = \{(u(t), v(t)) \,/ u(t), v(t) \ \textit{are measurable}, \\ &0 \le u_{\min} \le u(t) \le u_{\max} \le 1 \ \textit{and} \\ &0 \le u_{\min} \le v(t) \le v_{\max} \le 1, \quad t \in [0, t_f]\} \end{aligned} \tag{15}$$

that indicates an admissible control set. Now, we consider an optimal control problem to minimize the objective functional subject to system (14).

$$J(u, v) = U(t_f) + U_p(t_f) + \int_0^{t_f} [K_1 U(t) + K_2 U_P(t) + \frac{1}{2} K_3 u^2(t) + \frac{1}{2} K_4 v^2(t)] dt. \tag{16}$$

The system model with Controls Dynamics equation (14). Here K_1 and K_2 are positive constants to keep a balance in the size of $U(t)$ and $U_P(t)$, respectively. The square of the control variable reflects the severity of the side effects of awareness programs. In the objective functional, K_3 and K_4 are a positive weight parameter which are associated with the control $u(t)$ and $v(t)$ respectively. t_f is the final time. The objective of our work is to minimize the number of unemployed people and the number of unemployed beneficiaries of government programs individuals and to maximize the total number of employed individuals by using possible minimal control variables $u(t)$ and $v(t)$.

In other words, we seek the optimal controls $u^*(t)$ and $v^*(t)$ such that

$$J(u^*, v^*) = \min_{(u,v)\in\Omega_{ad}} \left(J(u, v)\right),$$

where Ω_{ad} is the set of admissible controls defined by (15).

6.2.1 Existence of an Optimal Control

Theorem 3. *Consider the control problem with system (14). There exists an optimal control* $(u^*, v^*) \in U_{ad}$ *such that*

$$J(u^*, v^*) = \min_{(u,v)\in\Omega_{ad}} \left(J(u, v)\right).$$

Proof. The existence of the optimal control can be obtained using a result by Fleming & Rishel (2012), checking the following steps:

- It follows that the set of controls and corresponding state variables is nonempty, we will use a simplified version of an existence result (Boyce et al., 2021), Theorem 7.1.1.
- $J(u, v)$ is convex in Ω_{ad}.
- The control space Ω_{ad} is convex and closed by definition.
- All the right hand sides of equations of system are continuous, bounded above by a sum of bounded control and state, and can be written as a linear function of u, v with coefficients depending on time and state.
- The integrand in the objective functional $K_1U(t) + K_2U_P(t) + \frac{1}{2}K_3u^2(t) + \frac{1}{2}K_4v^2(t)$ is clearly convex on Ω_{ad}.
- There exists constants $\rho_1, \rho_2, \rho_3 > 0$ and ρ such that$K_1U(t) + K_2U_P(t) + \frac{1}{2}K_3u^2(t) + \frac{1}{2}K_4v^2(t)$ satisfies: $K_1U(t) + K_2U_P(t) + \frac{1}{2}K_3u^2(t) + \frac{1}{2}K_4v^2(t) \geqslant -\rho_1 + \rho_2|u|^\rho + \rho_3|v|^\rho$.

The state variables being bounded, let $\rho_1 = \sup_{t\in[0,t_f]} \left(K_1U(t) + K_2U_P(t)\right), \rho_2 = \dfrac{K_3}{2}, \rho_3 = \dfrac{K_4}{2}$ and $\rho = 2$ then it follows that:

$$K_1U(t) + K_2U_P(t) + \frac{1}{2}K_3u^2(t) + \frac{1}{2}K_4v^2(t) \geqslant -\rho_1 + \rho_2|u|^\rho + \rho_3|v|^\rho$$

Then from Fleming & Rishel (2012) we conclude that there exists an optimal control. □

6.2.2 Characterization of an optimal control

In order to derive the necessary conditions for the optimal control, we apply Pontryagin maximum principle to the Hamiltonian H at time t defined by

$$H = K_1U(t) + K_2U_P(t) + \frac{1}{2}K_3u^2(t) + \frac{1}{2}K_4v^2(t)) + \sum_{i=1}^{4}\left(\lambda_i(t)f_i(U, E, U_p, P)\right), \qquad (17)$$

where f_i, is the right-hand side of the differential equation of the i^{th} state variable in system (14).

Theorem 4. *Given an optimal control $(u^*, v^*) \in \Omega_{ad}$ and the solutions $U^*(t), E^*(t), U_p^*(t)$ and $P^*(t)$ of the corresponding state system 14, there exists adjoint functions $\lambda_1(t), \lambda_2(t), \lambda_3(t)$ and $\lambda_4(t)$ satisfying:*

$$\begin{aligned}
\frac{d\lambda_1(t)}{dt} &= K_1 + (1+d)\lambda_1(t) + (\beta E(t)(1+u(t)))(\lambda_1(t) - \lambda_2(t)) + \lambda P(t)(\lambda_1(t) - \lambda_3(t)) \\
\frac{d\lambda_2(t)}{dt} &= (1+d+\alpha)\lambda_2(t) + ((\beta U(t)(1+u(t)) - \gamma))(\lambda_1(t) - \lambda_2(t)) - \mu\lambda_4(t)) \\
\frac{d\lambda_3(t)}{dt} &= K_2 + (1+d)\lambda_3(t) + \lambda_0(\lambda_3(t) - \lambda_1(t)) - (a + v(t))(\lambda_3(t) - \lambda_2(t)) \\
\frac{d\lambda_4(t)}{dt} &= (1+\mu_0)\lambda_4(t) + \lambda U(t)(\lambda_1(t) - \lambda_3(t))
\end{aligned} \tag{18}$$

with the following transversality conditions at final time t_f,

$$\lambda_1(t_f) = 0, \lambda_2(t_f) = 0, \lambda_3(t_f) = 0, \text{ and } \lambda_4(t_f) = 0. \tag{19}$$

Furthermore, we may characterize the optimal pair by the piecewise continuous functions

$$u^*(t) = min\left[u_{max}, max\left(u_{min}, \frac{1}{K_3}(\lambda_1 - \lambda_2)U^*\right)\right], \tag{20}$$

$$v^*(t) = min\left[v_{max}, max\left(v_{min}, \frac{1}{K_4}(\lambda_1 - \lambda_3)U_p^*\right)\right]. \tag{21}$$

Proof. The Hamiltonian is defined as follows

$$\begin{aligned}
&H(U, E, U_p, P, u, v, \lambda) = K_1U(t) + K_2U_P(t) + \tfrac{1}{2}K_3u^2(t) + \tfrac{1}{2}K_4v^2(t) \\
&+\lambda_1[A - \beta(1+u(t))U(t)E(t) - \lambda U(t)P(t) + \gamma E(t) + \lambda_0 U_P(t) - dU(t)] \\
&+\lambda_2[\beta(1+u(t))U(t)E(t) - \gamma E(t) + aU_P(t) - \alpha E(t) - dE(t) + v(t)U_P(t)] \\
&+\lambda_3[\lambda U(t)P(t)) - \lambda_0 U_P(t) - aU_P(t) - dU_P(t) - v(t)U_P(t)]. \\
&+\lambda_4[\mu E(t) - \mu_0 P(t)].
\end{aligned} \tag{22}$$

Here, $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in I\!R^4$ is an adjoint variable. The optimal system of equations is found by taking the appropriate partial derivatives of Hamiltonian 23 with respect to the associated state variables.

On the base of Pontryagin's Maximum Principale, the adjoint equations and corresponding final time conditions (transversality conditions) are given:

$$\begin{cases}
\lambda_1' \frac{\partial H}{\partial U}, & \lambda_1(t_f) = 0 \\
\lambda_2' \frac{\partial H}{\partial E}, & \lambda_2(t_f) = 0 \\
\lambda_3' \frac{\partial H}{\partial U}, & \lambda_3(t_f) = 0 \\
\lambda_3' \frac{\partial H}{\partial P}, & \lambda_4(t_f) = 0
\end{cases}$$

for $t \in [0, t_f]$; the optimal paer control $u^*(t)$ and $v^*(t)$ are obtained as well

$$\begin{aligned}
\frac{\partial H(t)}{\partial u(t)} &= 0 \\
&\textit{and} \\
\frac{\partial H(t)}{\partial v(t)} &= 0
\end{aligned}$$

$$\begin{cases} \dfrac{\partial H}{\partial u} = K_3 u - \beta U E \lambda_1 + \beta U E \lambda_2 = 0 \\ \\ \dfrac{\partial H}{\partial v} = K_4 v - U_p \lambda_2 + U_p \lambda_3 = 0 \end{cases} \tag{23}$$

we have

$$\begin{cases} u(t) &= \dfrac{1}{K_3}(\lambda_1 - \lambda_2)\beta U E \\ \\ v(t) &= \dfrac{1}{K_4}(\lambda_2 - \lambda_3) U_p. \end{cases} \tag{24}$$

By the bounds in Ω_{ad} of the controls, it is easy to obtain u^* and v^* are given in (21) and (22) the form of system in the form of system (14). □

We point out that the optimal system consists of the state system (14) with the initial conditions, adjoint system (20) with transversality conditions (20), and optimality condition (20)-(21). Thus, we have the following optimal system

$$\begin{cases} \frac{dU(t)}{dt} = A - \beta(1+u(t))U(t)E(t) - \lambda U(t)P(t) + \gamma E(t) + \lambda_0 U_P(t) - dU(t). \\ \\ \frac{dE(t)}{dt} = \beta(1+u(t))U(t)E(t) - \gamma E(t) + aU_P(t) - \alpha E(t) - dE(t) + v(t)U_P(t). \\ \\ \frac{dU_P(t)(t)}{dt} = \lambda U(t)P(t)) - \lambda_0 U_P(t) - aU_P(t) - dU_P(t) - v(t)U_P(t). \\ \\ \frac{dP(t)}{dt} = \mu E(t) - \mu_0 P(t), \\ \\ \dfrac{d\lambda_1(t)}{dt} = K_1 + (1+d)\lambda_1(t) + (\beta E(t)(1+u(t)))(\lambda_1(t) - \lambda_2(t)) + \lambda P(t)(\lambda_1(t) - \lambda_3(t)) \\ \\ \dfrac{d\lambda_2(t)}{dt} = (1+d+\alpha)\lambda_2(t) + ((\beta U(t)(1+u(t)) - \gamma))(\lambda_1(t) - \lambda_2(t))(-\mu\lambda_4(t)) \\ \\ \dfrac{d\lambda_3(t)}{dt} = K_2 + (1+d)\lambda_3(t) + \lambda_0(\lambda_3(t) - \lambda_1(t)) - (a+v(t))(\lambda_3(t) - \lambda_2(t)) \\ \\ \dfrac{d\lambda_4(t)}{dt} = (1+\mu_0)\lambda_4(t) + \lambda U(t)(\lambda_1(t) - \lambda_3(t)). \\ \\ U(0), E(0), U_p(0), P(0) \geqslant 0, \\ \\ \lambda_i(t_f) = 0, \quad i = 1,2,3,4. \end{cases} \tag{25}$$

where

$$u(t) = min\left[u_{max}, max\left(u_{min}, \frac{1}{K_3}(\lambda_1 - \lambda_2)\beta U E\right)\right],$$

$$v(t) = min\left[v_{max}, max\left(v_{min}, \frac{1}{K_4}(\lambda_2 - \lambda_3) U_p\right)\right].$$

6.3 Numerical simulations for the optimal control problem

In this part, our numerical results concern the comparison between the model with control strategy and the model without control. Figures 9 -12 show the evolution of the different classes without and with control.

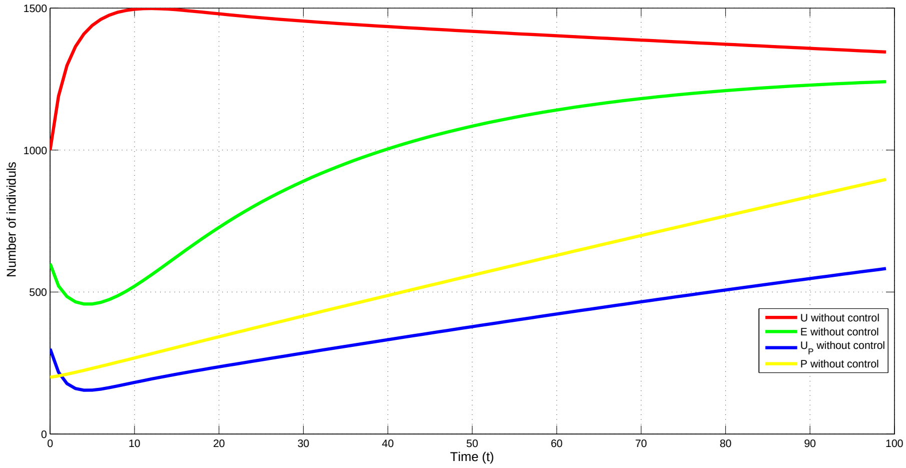

Figure 9: Evolution of the different classes without control over time

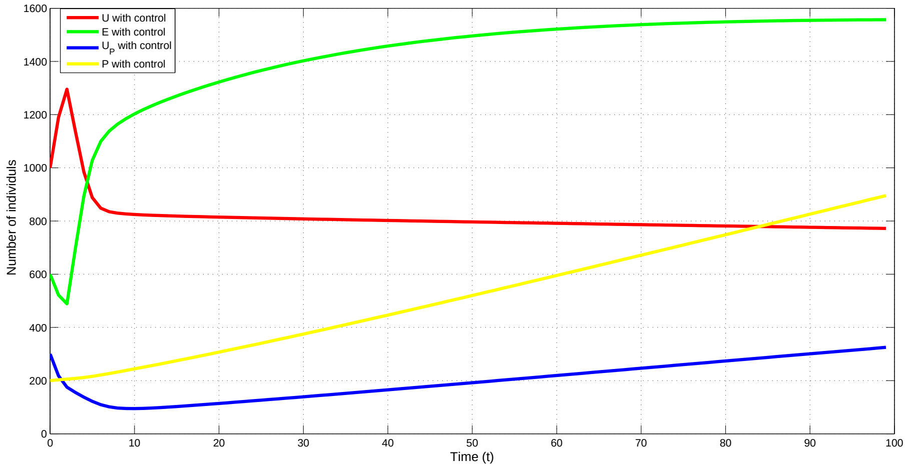

Figure 10: Evolution of the different classes with control as a function of time

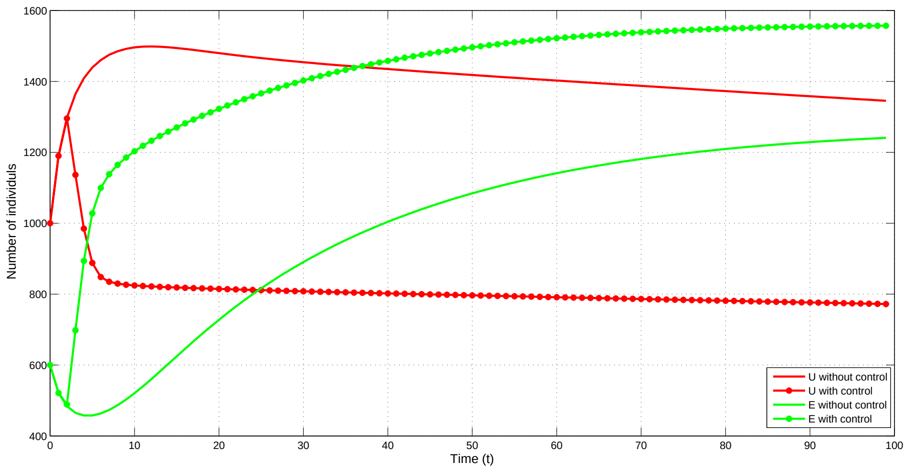

Figure 11: Variation in unemployment $U(t)$ and employment $E(t)$ without and with control

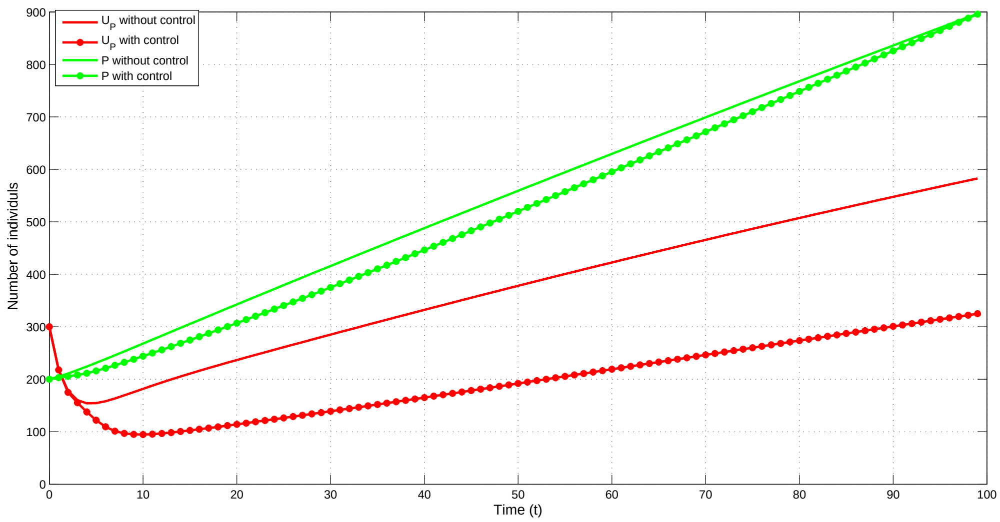

Figure 12: Variation of $U_P(t)$ and $P(t)$ without and whit control

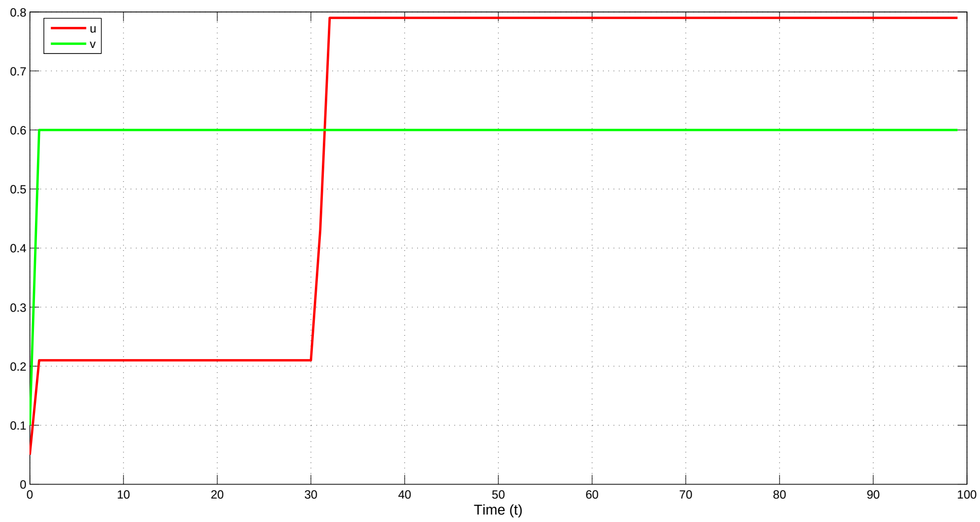

Figure 13: Numerical illustration of the control variables

We can clearly see that, with a control strategy, there will be fewer unemployed people and more people benefiting from the trainee-ship who are integrated into the labour market than in the absence of control measures. This means that the role of control strategies is to reduce unemployment, which can help to avoid the negative effects of this phenomenon on the population and the economy.

7 Conclusion

In this research, a new mathematical model has been developed to investigate how internships programs affects the reduction of unemployment. It is assumed that the government implements integration programs to help the unemployed improve their abilities and familiarise themselves with the workplace in order to integrate them. The theory of differential equation stability has been used to investigate the provided model. In the results is the identification of two points of equilibrium.These points display local and global asymptotic stability with certain conditions Moreover, numerical illustrations are given to confirm the theoretical findings. To find out further about this study, a two-variable optimal control approach is used to optimize the performance and efficiency of government initiatives. Our analysis also shows that developing the skills of the unemployed through work placements makes a significant contribution to reducing the number of unemployed. Additionally, In order to achieve effective government strategies, the efforts must be concentrated on creating jobs and subsidising companies to integrate unemployed people on work placements. the efforts must be focused on creating more jobs and subsidising firms to integrate unemployed people taking part in work placements.

For further research we can extend our model in more realistic framework and uncertain environment using stochastic modeling approach (see, Akdim et al. (2021); Hajri et al. (2024)). Also, it will be interesting to study the dynamics and control of other macroeconomics phenomena as inflation rate (see for instance Bikourne et al. (2023)) or public debt control Stein (2004).

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